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Correlation function of random-field Ising chains: is it Lorentzian or not?

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Abstract. The two-point connected correlation function, or wavevector-dependent susceptibility, of the ferromagnetic Ising chain in a random field is calculated exactly at any temperature, for a two-parameter family of diluted symmetric exponential distributions of the magnetic fields. Thermodynamic properties of this model have been derived in a previous work by the authors. Besides the correlation function itself, the solution provides exact results for the (usual) susceptibility, the correlation length and the Edwards-Anderson parameter. The low-temperature regime is examined in full detail: we obtain in closed form the limit value at zero temperature of various quantities, and the first correction to this limit, which behaves linearly with temperature. The correlation length is discontinuous at zero temperature. We also derive the scaling form of the correlation function for small p , where the dilution p is the probability for a spin to be subjected to a non-zero random field. Even in this limit, the correlation function is more complicated than the simple Lorentzian predicted, e.g., by mean-field theory.

1. Introduction

The Ising model in a random magnetic field has been the subject of much recent work. Its main interest originates in the presence of both frustration and randomness. There can therefore exist numerous metastable states, just as in spin glasses, which bring much difficulty in the analysis of both thermodynamic and dynamical properties. Experimental realisations of random-field Ising models are provided by diluted anti-ferromagnets [1]. The usual fitted form of their structure factor (correlation function) is the superposition of a Lorentzian and a Lorentzian squared (see e.g. [2]).

Even in one dimension, the random-field Ising model is exactly soluble only for very few particular distributions of the random magnetic fields h_n . The symmetric binary distribution ($h_n = \pm H$) has been considered by Derrida and co-workers [3], who have evaluated the energy and entropy at zero temperature. Grinstein and Mukamel [4] have calculated both the thermodynamic properties and the two-point correlation functions, at any temperature, for a distribution, given below in equation (1.4), where the fields are either zero, or plus or minus infinity. We also mention several related works. The limit where the ferromagnetic coupling J is much larger

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than the random fields has been investigated by Derrida and Hilhorst [5]. It has also been shown that some fluctuating quantities, such as the local fields, may have a fractal or Cantor-like support [6–8]. More recently, the energy and two-point correlation function of a one-dimensional lattice gas model in a random potential at zero temperature have also been investigated [9].

In a previous publication [10], the authors have analysed the thermodynamic properties at any temperature for exponential distributions of the random fields, both symmetric and non-symmetric, and with dilution. The method used has been introduced by one of us [11, 12]. It can be applied to a large class of disordered one-dimensional systems, and yields an exact solution for a specific form of the distribution (usually exponential) of the random variables of the problem. Diluted randomness [13], as well as random interactions with short-range correlations [14], can also be treated by this approach. The correlation functions of some linear problems, such as the tight-binding Schrödinger equation or harmonic chains with random masses, can also be determined using the same approach [12, 13]. In this paper, we present the calculation of the two-point function $\chi(q)$ of the random-field Ising model considered in [10]. We first need to recall the definition of this quantity.

For each realisation of the random fields h_n , the usual correlation functions, or thermal averages, denoted by brackets, such as $\langle \sigma_n \rangle$ or $\langle \sigma_m \sigma_n \rangle$, depend on all the fields. An ensemble averaging over the distribution of the random fields, denoted as usual by a horizontal bar, is necessary to produce well behaved translationally invariant correlation functions.

We will restrict ourselves to the case of the two-point correlation function. There are two inequivalent ways of defining such a quantity, namely

- (1) the connected correlation function $\chi(q)$, defined by

$$\chi(q) = \sum_n \chi_n e^{inq} \quad \text{with} \quad \chi_n = \overline{\langle \sigma_n \sigma_0 \rangle} - \overline{\langle \sigma_n \rangle} \overline{\langle \sigma_0 \rangle} \quad (1.1a)$$

- (2) the full correlation function $S(q)$, defined by

$$S(q) = \sum_n S_n e^{inq} \quad \text{with} \quad S_n = \overline{\langle \sigma_n \sigma_0 \rangle}. \quad (1.1b)$$

The first of these quantities is a generalisation of the susceptibility, and characterises the response of the system to a modulated external perturbation. Hence $\chi(q)$ is also termed the wavevector-dependent susceptibility. The second definition corresponds to the quantity measured in scattering experiments. $S(q)$ is also referred to as the magnetic structure factor.

The correlation χ_0 at coinciding points is related to the well known Edwards-Anderson order parameter

$$\int_0^{2\pi} \frac{dq}{2\pi} \chi(q) = \chi_0 = 1 - q_{EA} \quad \text{with} \quad q_{EA} = \overline{\langle \sigma_0 \rangle^2}. \quad (1.2)$$

The aim of this paper is to derive the exact expression of the correlation function $\chi(q)$ for the specific class of random field distributions already considered in [10]. Setting

$$h_n = Hx_n \quad H > 0, \quad -\infty < x_n < +\infty \quad (1.3a)$$

the distribution $\tilde{R}(x_n)dx_n$ of the dimensionless reduced fields x_n has the following

‘diluted symmetric exponential’ form:

$$\tilde{R}(x) = \frac{p}{2} e^{-|x|} + r\delta(x) \quad \text{with} \quad r = 1 - p. \tag{1.3b}$$

The dilution probability p and the field strength H are two arbitrary parameters.

In the $H = \infty$ limit, the above distribution simplifies. Indeed, if the variable x_n vanishes, so does the field h_n . If $x_n \neq 0$, h_n is infinite in absolute value, with the sign of x_n . Hence the distribution of the random field reads

$$h_n = \begin{cases} -\infty & \text{with probability } p/2 \\ 0 & \text{with probability } r \\ +\infty & \text{with probability } p/2. \end{cases} \tag{1.4}$$

As recalled above, this limit case has been studied, using a direct enumeration approach, by Grinstein and Mukamel, in [4].

The set-up of this paper is as follows. In § 2, we present some general formalism, showing in particular how the correlation function $\chi(q)$ can be evaluated as a susceptibility, namely by differentiating the free energy w.r.t ‘source’ terms, to be added to the magnetic fields. Section 3 is devoted to the exact calculation at any temperature of the correlation function for the distribution (1.3) of the random fields. Its first part presents an alternative derivation of the free energy, already studied in [10]. The final result involves the solution of coupled difference equations for six sequences. Section 4 contains a detailed analysis of the low-temperature behaviour of the correlation function. We consider in particular quantities such as the susceptibility, the correlation length, and the Edwards–Anderson order parameter. The final results are given in § 5: for each quantity, we obtain in closed form both the $T = 0$ limit value, and the first correction to it, usually linear in temperature. Section 6 presents a discussion.

2. Generalities

The purpose of this section is to present the general formalism needed for the evaluation of the correlation function $\chi(q)$ as a susceptibility.

We consider the one-dimensional Ising model in a random field, with the Hamiltonian

$$\mathcal{H} = -J \sum_n \sigma_n \sigma_{n+1} - \sum_n h_n \sigma_n. \tag{2.1}$$

We restrict ourselves to the ferromagnetic case $J > 0$. The fields h_n are independent random variables, with a common probability distribution $R(h)dh$. We first recall briefly the transfer matrix formalism for the free energy of the model. The partition function Z_N at temperature $T = 1/\beta$ of a finite chain having N sites with periodic boundary conditions is given by

$$Z_N = \text{tr} \prod_{n=1}^N T_n \quad T_n = \begin{pmatrix} e^{\beta J + \beta h_n} & e^{-\beta J + \beta h_n} \\ e^{-\beta J - \beta h_n} & e^{\beta J - \beta h_n} \end{pmatrix}. \tag{2.2}$$

The quenched free energy of the model is therefore equal to the Lyapunov exponent of the infinite matrix product

$$-\beta F = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \text{tr} \prod_{n=1}^N T_n. \tag{2.3}$$

A convenient way to estimate this quantity is to introduce a sequence of vectors (x_n, y_n) such that (x_n, y_n) is the image by T_n of (x_{n-1}, y_{n-1}) , and to consider the ratios $\rho_n = x_n / y_n$, which obey

$$\rho_n = e^{2\beta h_n} \frac{e^{2\beta J} \rho_{n-1} + 1}{\rho_{n-1} + e^{2\beta J}}. \tag{2.4}$$

When n becomes large, the distribution of $\rho_n > 0$ reaches a stationary limit, invariant under the transformation (2.4). The free energy is then given by

$$-\beta F = -\beta J - \beta \bar{h} + \langle \ln(\rho_n + e^{2\beta J}) \rangle. \tag{2.5}$$

From now on, we focus our attention onto symmetric field distributions: $R(h) = R(-h)$. Then the average field \bar{h} vanishes, and it is advantageous to perform the change of variable

$$\rho_n = \frac{1 - z_n}{1 + z_n} \quad z_n = \frac{1 - \rho_n}{1 + \rho_n} \tag{2.6}$$

which maps the domain $\rho_n > 0$ onto $-1 < z_n < +1$. The transformation (2.4) is now

$$z_n = \frac{\tau z_{n-1} - t_n}{1 - \tau z_{n-1} t_n} \tag{2.7}$$

and the free energy is given by

$$-\beta F = \ln(2 \cosh \beta J) + \left\langle \ln \frac{1 + \tau z_n}{1 + z_n} \right\rangle \tag{2.8}$$

with the notation

$$\tau = e^{-2\mu} = \tanh \beta J \quad t_n = \tanh \beta h_n. \tag{2.9}$$

The easiest way of evaluating the correlation function $\chi(q)$ is to add the following deterministic ‘sources’ to the random fields

$$\beta h_n \rightarrow \beta h'_n = \beta h_n + \eta_n \quad \eta_n = X e^{inq} + Y e^{-inq}. \tag{2.10}$$

Let z'_n denote the variables associated with the modified fields h'_n , and F' be the corresponding free energy. It is then easily shown that we have

$$\chi(q) = \left(- \frac{\partial^2 \beta F'}{\partial X \partial Y} \right)_{X=Y=0}. \tag{2.11}$$

In order to obtain for this quantity an expression analogous to (2.8), the expansion of (2.7) up to second order in the η_n is needed. Let $z'_n = z_n + u_n + v_n + \dots$, where z_n is the value in the absence of the sources η_n , u_n is linear in the sources, v_n is quadratic in them, etc. A simple but lengthy computation yields

$$u_n = \frac{1 - t_n^2}{(1 - \tau z_{n-1} t_n)^2} [\tau u_{n-1} + (\tau^2 z_{n-1}^2 - 1) \eta_n] \tag{2.12a}$$

$$v_n = \frac{1 - t_n^2}{(1 - \tau z_{n-1} t_n)^3} [\tau(1 - \tau z_{n-1} t_n) v_{n-1} + \tau^2 t_n u_{n-1}^2 + 2\tau(\tau z_{n-1} - t_n) u_{n-1} \eta_n + (\tau^2 z_{n-1}^2 - 1)(\tau z_{n-1} - t_n) \eta_n^2]. \tag{2.12b}$$

If we now introduce the following expansion in X and Y :

$$u_n = X e^{inq} u_n^+ + Y e^{-inq} u_n^- \tag{2.13a}$$

$$v_n = X^2 e^{2inq} v_n^+ + 2XY v_n^0 + Y^2 e^{-2inq} v_n^- \tag{2.13b}$$

the five random variables $u_n^+, u_n^-, v_n^+, v_n^-, z_n$ have stationary distributions as n becomes large, just as ρ_n or z_n . Moreover, (2.11) can be rewritten as

$$\chi(q) = \left\langle 2v_n^0 \left(\frac{\tau}{1 + \tau z_n} - \frac{1}{1 + z_n} \right) + u_n^+ u_n^- \left(\frac{1}{(1 + z_n)^2} - \frac{\tau^2}{(1 + \tau z_n)^2} \right) \right\rangle. \tag{2.14}$$

Hence v_n^+ and v_n^- are not needed to evaluate the correlation function. Moreover, if we consider only real values of the wavevector q , u_n^+ is the complex conjugate of u_n^- . Hence we are left with two random variables, namely u_n^- and v_n^0 , which obey the recursion relations

$$u_n^- = \tau e^{iq} \varphi_n u_{n-1}^- + z_n^2 - 1 \tag{2.15a}$$

$$v_n^0 = \varphi_n [\tau v_{n-1}^0 + \tau^2 \gamma_n u_{n-1}^+ u_{n-1}^- + \tau z_n (e^{iq} u_{n-1}^- + \text{cc}) + (\tau^2 z_{n-1}^2 - 1) z_n] \tag{2.15b}$$

with the notation

$$\varphi_n = \frac{1 - t_n^2}{(1 - \tau z_{n-1} t_n)^2} \quad \gamma_n = \frac{t_n}{1 - \tau z_{n-1} t_n}. \tag{2.16}$$

Here and in the following, cc denotes the complex conjugate of an expression.

A last comment on the above formalism is in order. The RHS of (2.14) seems to be singular as $z_n \rightarrow -1$, but no singularity at all is actually present. Indeed (2.7) implies that $(1 + z_n)$ is proportional to $(t_n - 1)$, for fixed z_{n-1} . The quantity v_n^0 brings one factor of $(t_n - 1)$ in the numerator through φ_n , and u_n^+ and u_n^- bring two such factors. These cancel exactly the factors of $(1 + z_n)$ in the denominators. This remark will be useful in the next section.

Equations (2.14)–(2.16) will be the starting point of our exact calculation, presented in § 3, of the correlation function for the diluted symmetric exponential distribution (1.3) defined in § 1. We end up this section by illustrating the above formalism in the (trivial) case where there is no random field. Then we have $\langle \sigma_n \rangle = 0$, $\langle \sigma_m \sigma_n \rangle = \tau^{|m-n|}$, and no averaging over disorder is needed. Hence both correlation functions are

$$S(q) = \chi(q) = \frac{1 - \tau^2}{1 + \tau^2 - 2\tau \cos q}. \tag{2.17}$$

This result for $\chi(q)$ is easily recovered from (2.14), since it follows from (2.7) and (2.15) that the stationary values of the above-defined variables are $z_n = 0$, $u_n^- = -(1 - \tau e^{iq})^{-1}$ and $v_n = 0$.

3. Exact solution at finite temperature

This section is devoted to an exact determination of the correlation function $\chi(q)$ for the distribution (1.3) of the random magnetic fields. We will use the general formalism introduced in § 2, and begin with a determination of the free energy of the model. The present approach is slightly different from that of [10], where non-symmetric field distributions were considered as well. For both the free energy and the correlation length, the key point is to integrate exactly over the random fields. This method yields three-term recursion relations for six unknown sequences. A very analogous structure is met in the study of the response functions of some linear problems, such as harmonic chains and tight-binding electrons [12, 13].

3.1. *The free energy*

The free energy is obtained by expanding (2.8) as

$$-\beta F = \ln(2 \cosh \beta J) + \sum_{k \geq 2, \text{even}} \frac{1 - \tau^k}{k} c_k \tag{3.1}$$

where c_k denote the moments

$$c_k = \langle z_n^k \rangle \tag{3.2}$$

the average being performed WRT the stationary distribution of z_n . It follows from (2.7) that, if the sign of all the random fields h_n is changed, so is the sign of the z_n . Hence, if the random field distribution is even, so is the distribution of the variable z_n , and the moments c_k vanish for odd values of k . This property has been used in the derivation of (3.1). More generally, all the quantities to be considered hereafter will have analogous simple parity properties.

The c_k can actually be derived from the recursion relation (2.7) by means of the following method, which will be used extensively hereafter. Consider $z_n(x_n)$ as a function of the random variable x_n , through the notation (1.3a), at fixed z_{n-1} . One has then

$$\frac{\partial z_n}{\partial x_n} = \frac{\lambda}{2} (z_n^2 - 1) \tag{3.3}$$

with the notation

$$\lambda = 2\beta H. \tag{3.4}$$

Introduce now the integrals

$$I_k(z_{n-1}) = \int \frac{e^{-|x|} dx}{2} z_n^k(x). \tag{3.5}$$

It is easily shown, by means of two integrations by parts using (3.3), that these quantities obey

$$I_k = (\tau z_{n-1})^k + \frac{\lambda^2}{4} k[(k+1)I_{k+2} + (k-1)I_{k-2} - 2kI_k]. \tag{3.6}$$

Let now C_k denote the averages

$$C_k = \langle I_k(z_{n-1}) \rangle \tag{3.7a}$$

WRT the stationary distribution of z_{n-1} . The C_k can also be viewed as the following 'restricted moments':

$$C_k = \langle z_n^k \rangle_{x_n \neq 0} \tag{3.7b}$$

where the average over the last random variable x_n is performed WRT the exponential component $e^{-|x_n|} dx_n/2$ of the distribution (1.3b), rather than WRT the full 'diluted' distribution.

The definition (3.5) and the recursion (3.6) lead to the following coupled recursive equations:

$$c_k = \tau \tau^k c_k + p C_k \tag{3.8a}$$

$$C_k = \tau^k c_k + \frac{\lambda^2}{4} k[(k+1)C_{k+2} + (k-1)C_{k-2} - 2kC_k]. \tag{3.8b}$$

More precisely, (3.8a) is a simple consequence of the existence of two components in the distribution (1.3), and the definition (3.7b), and (3.8b) follows directly from (3.6). Equation (3.8) can be equivalently rewritten as

$$c_k = \frac{p}{1 - r\tau^k} C_k \tag{3.9a}$$

$$\frac{\lambda^2}{4} k[(k+1)C_{k+2} + (k-1)C_{k-2} - 2kC_k] = \frac{1 - \tau^k}{1 - r\tau^k} C_k. \tag{3.9b}$$

The parity of the distribution of z_n implies that the averages C_k vanish for odd values of k , just as c_k . It follows from their definition that $c_0 = C_0 = 1$. Moreover, since the variables z_n lies between -1 and $+1$, and its distribution contains no delta function at ± 1 , both sequences c_k and C_k go to zero as $k \rightarrow \infty$. On the other hand, the large- k behaviour of the general solution of (3.9) can be extracted by replacing the difference equation (3.9b) by the following differential equation:

$$C = \lambda^2 \left(k^2 \frac{d^2 C}{dk^2} + k \frac{dC}{dk} \right). \tag{3.10}$$

This equation implies that two independent solutions of (3.9b) behave as $k^{1/\lambda}$ and $k^{-1/\lambda}$. Therefore the condition that c_k and C_k go to zero as $k \rightarrow \infty$ selects the decreasing solution, and the above set of boundary conditions determines a unique solution to (3.9).

The expression (3.1) of the free energy can be largely simplified by means of (3.9b) as

$$-\beta F = \ln(2 \cosh \beta J) + p \frac{\lambda^2}{4} (1 - C_2). \tag{3.11}$$

This result coincides with that given in [10], where it was obtained in a slightly different way. We mention that the sequence C_k was denoted there by d_k , and H by H_r .

3.2. The correlation function

By analogy with the previous subsection, we first expand the expression (2.14) of the correlation function $\chi(q)$ as

$$\chi(q) = \sum_{k \geq 2, \text{even}} (1 - \tau^k) g_k \tag{3.12}$$

with the definitions

$$g_k = (k-1)e_k + 2f_k \quad e_k = \langle u_n^+ u_n^- z_n^{k-2} \rangle \quad f_k = \langle v_n^0 z_n^{k-1} \rangle. \tag{3.13}$$

In order to evaluate these quantities, we follow the lines of the calculation of the free energy exposed in the previous subsection. Consider the variables φ_n and γ_n defined in (2.16) as functions of x_n , at fixed z_{n-1} . One has then

$$\frac{\partial \varphi_n}{\partial x_n} = \lambda z_n \varphi_n \quad \frac{\partial \gamma_n}{\partial x_n} = \frac{\lambda}{2} \varphi_n. \tag{3.14}$$

We are thus led to define the integrals

$$J_{k,l}(z_{n-1}) = \int \frac{e^{-|x|} dx}{2} z_n^{k-l}(x) \varphi'(x) \tag{3.15a}$$

$$K_k(z_{n-1}) = \int \frac{e^{-|x|} dx}{2} z_n^{k-1}(x) \varphi_n(x) \gamma_n(x). \tag{3.15b}$$

Two integrations by parts using (3.14) then yield

$$J_{k,l} = (\tau z_{n-1})^{k-l} + \frac{\lambda^2}{4} [(k+l)(k+l+1)J_{k+2,l} + (k-l)(k-l-1)J_{k-2,l} - 2(k^2-l(l-1))J_{k,l}] \tag{3.16a}$$

$$K_k = \frac{\lambda^2}{4} [(k+1)(k+2)K_{k+2} + (k-1)(k-2)K_{k-2} - 2k^2K_k + 2(k+2)J_{k+2,2} - 2(k-1)J_{k,2}]. \tag{3.16b}$$

We now define the averages

$$d_k = \langle u_n^- z_n^{k-1} \rangle \tag{3.17a}$$

$$D_k = \tau \langle u_{n-1}^- J_{k,1}(z_{n-1}) \rangle = \tau \langle u_{n-1}^- z_n^{k-1} \varphi_n \rangle_{x_n \neq 0} \tag{3.17b}$$

$$E_k = \tau^2 \langle u_{n-1}^+ u_{n-1}^- J_{k,2}(z_{n-1}) \rangle = \tau^2 \langle u_{n-1}^+ u_{n-1}^- z_n^{k-2} \varphi_n^2 \rangle_{x_n \neq 0} \tag{3.17c}$$

$$F_k = \tau \langle v_{n-1}^0 J_{k,1}(z_{n-1}) \rangle + \tau^2 \langle u_{n-1}^+ u_{n-1}^- K_k(z_{n-1}) \rangle \\ = \langle \tau v_{n-1}^0 z_n^{k-1} \varphi_n + \tau^2 u_{n-1}^+ u_{n-1}^- z_n^{k-1} \varphi_n \gamma_n \rangle_{x_n \neq 0} \tag{3.17d}$$

$$G_k = (k-1)E_k + 2F_k. \tag{3.17e}$$

It turns out that the separate evaluation of e_k , E_k , f_k , and F_k will not be needed. These quantities will indeed only show up hereafter through the combinations g_k and G_k . Equation (3.16) implies the following coupled recursive relations for the four quantities d_k , D_k , g_k , and G_k :

$$d_k = c_{k+1} - c_{k-1} + e^{iq}(r\tau^k d_k + pD_k) \tag{3.18a}$$

$$D_k = \tau^k d_k + \frac{\lambda^2}{4} [(k+1)(k+2)D_{k+2} + (k-1)(k-2)D_{k-2} - 2k^2 D_k] \tag{3.18b}$$

$$g_k = r\tau^k g_k + pG_k + S_{k+1} - S_{k-1} \tag{3.18c}$$

$$G_k = \tau^k g_k + \frac{\lambda^2}{4} [(k+1)(k+2)G_{k+2} + (k-1)(k-2)G_{k-2} - 2k^2 G_k] \tag{3.18d}$$

with the notation

$$S_k = k(d_k + c_k + c_{k-1} - c_{k+1}). \tag{3.18e}$$

The large- k behaviour of the general solution of the above equations can be derived by going to differential equations, analogous to (3.10). This procedure shows that two independent solutions for the four sequences kd_k , kD_k , kg_k , and kG_k , behave as $k^{1/\lambda}$ and $k^{-1/\lambda}$. On the other hand, the remark below (2.16) on the regularity of the expression (2.14) implies, after expansion in powers of z_n , that the four above mentioned sequences go to zero as $k \rightarrow \infty$. For instance, the finiteness of the quantity

$$\left\langle u_n^- \frac{1}{1+z_n} \right\rangle = \sum_{k \geq 1, \text{odd}} d_k \tag{3.19}$$

implies that d_k decays faster than $1/k$, i.e. kd_k goes to zero as $k \rightarrow \infty$. Hence the decreasing solution is always selected, and (3.18) determines d_k , D_k , g_k , and G_k in a unique way. Just as c_k and C_k , g_k and G_k are real, and vanish for odd values of k , while d_k and D_k are complex, and vanish for even values of k .

Finally, the expression (3.12) of the correlation function $\chi(q)$ can be simplified by means of (3.18c, d) as

$$\chi(q) = -p \frac{\lambda^2}{2} G_2 - S_1. \tag{3.20}$$

Just as in the case of some linear systems [12, 13], the correlation function can also be expressed in terms of c_k and d_k only. Indeed, by making use of (3.9) and (3.18), and noticing that the difference operator of (3.18d) is the adjoint of that of (3.8b), we can derive the following expression, quadratic in the c_k and d_k

$$\chi(q) = \sum_{k \geq 1, \text{odd}} (\tau^{k+1} c_{k+1} - \tau^{k-1} c_{k-1}) S_k. \tag{3.21}$$

To summarise this section, the evaluation of the correlation function $\chi(q)$ requires the following three steps. The appropriate boundary conditions have been discussed above.

- (1) Evaluation of c_k and C_k by solving (3.9). These real quantities vanish for odd k . The free energy F is then given by (3.11).
- (2) Evaluation of d_k and D_k by solving (3.18a, b). These complex quantities vanish for even k . The auxiliary sequence S_k is then given by (3.18e). This real quantity vanishes for even k .
- (3) Evaluation of g_k and G_k by solving (3.18c, d). These real quantities vanish for odd k . The correlation function $\chi(q)$ is then given by (3.20).

The above expressions can be checked in the absence of random fields (i.e. either $H = 0$, or $p = 0$). If $H = 0$, the only non-vanishing quantities are

$$\begin{aligned} c_0 = C_0 = 1 \quad d_1 = D_1 = \frac{-1}{1 - \tau e^{iq}} \quad S_1 = \frac{\tau^2 - 1}{1 - 2\tau \cos q + \tau^2} \\ g_2 = \frac{1}{1 - 2\tau \cos q + \tau^2} \quad G_2 = \frac{\tau^2}{1 - 2\tau \cos q + \tau^2}. \end{aligned} \tag{3.22}$$

If $p = 0$, the non-zero values of c_k , d_k , S_k , and g_k are still given by (3.22), but the sequences C_k , D_k , and G_k remain non-trivial. The expected result (2.17) for the correlation function is of course recovered in both cases.

3.3. The correlation length

The correlation length ξ is defined as the asymptotic decay rate of the correlation χ_n , defined in (1.1a)

$$\chi_n \underset{n \rightarrow \infty}{\sim} a_n e^{-n/\xi} \tag{3.23}$$

where the prefactor a_n has slower variations than any exponential, like e.g. a power law. Hence the quantity

$$\zeta = e^{-1/\xi} \tag{3.24}$$

is given by the location of the singularity of the correlation function $\chi(q)$ in the e^{iq} plane which has the largest modulus inside the unit circle. The nature of this singularity is related to the asymptotic behaviour of the prefactor a_n .

In the absence of random fields (i.e. $p = 0$ or $H = 0$), (2.17) yields

$$\zeta = \tau = \tanh \beta J. \tag{3.25}$$

In the $H = \infty$ limit, already studied in [4], we have

$$\zeta = r\tau \quad (3.26)$$

where r is the probability for the random magnetic field acting on any spin to vanish. Equations (3.25) and (3.26) show that, both for $H = 0$ and $H = \infty$, the correlation length has exponentially small corrections to its $T = 0$ value at low temperature. It will turn out that the situation is quite different for finite values of H .

For generic values of the parameters p and H , the function $\chi(q)$ is singular whenever the procedure summarised below (3.21) fails. This possibility originates in the boundary condition at infinity. The only equations which involve an e^{iq} dependence in their homogeneous terms are (3.18*a, b*) for the sequences d_k and D_k . Since the solution is determined by the condition that kd_k and kD_k go to zero as $k \rightarrow \infty$, the problem becomes ill defined when the generic solution of the linear homogeneous equation for D_k , namely

$$\frac{\zeta - \tau^k}{\zeta - r\tau^k} D_k = \frac{\lambda^2}{4} [(k+1)(k+2)D_{k+2} + (k-1)(k-2)D_{k-2} - 2k^2 D_k] \quad (3.27)$$

decreases for large k . Hence ζ is determined as the number with largest modulus inside the unit circle, such that (3.27) admits a solution which is both regular at $k = 1$, and decreasing at infinity. This solution is obviously defined up to a multiplicative constant. It turns out that ζ is real positive, and lies between the bounds (3.25) and (3.26) corresponding to $H = \infty$ and $H = 0$, respectively: $r\tau \leq \zeta \leq \tau$. Numerical data on the correlation length will be presented in § 5.2, after the study of the low-temperature limit of this quantity.

4. Low-temperature behaviour: technicalities

This section presents a detailed analysis of the low-temperature behaviour of the correlation function $\chi(q)$, starting from the exact expression, valid for any finite temperature, derived in § 3. We will follow the approach introduced in [10], which makes an extensive use of Laplace transforms.

The starting point of this analysis is the observation that all the non-vanishing moments and averages such as c_k , C_k , etc introduced in § 3 have a smooth and very slow k dependence at low temperature. More precisely, the typical value of k at which all the quantities have their relevant variations are of the order of $k_0 \sim 1/\mu \sim e^{2\beta J}$, with the notation of (2.9): this scale is exponentially large at low temperature. We are thus led to introduce the scaled variable

$$y = \frac{1}{\lambda} \ln 2k\mu = y_0 + \frac{1}{\lambda} \ln 2k \quad y_0 = -\frac{J}{H}. \quad (4.1)$$

The second equality is valid up to exponentially small terms, proportional to μ . Such corrections will be systematically forgotten in the following, since we only aim at deriving the value at zero temperature of the physical quantities, and the first low-temperature correction, usually linear in T . The various sequences will be replaced by functions of y , after appropriate rescaling. The difference equations for these sequences, derived in § 3, will become second-order differential equations, to be solved by the Laplace transform technique, taking special care of boundary conditions.

4.1. The function C and the free energy

For the sake of completeness and notational consistency, we first repeat the analysis of the low-temperature behaviour of the free energy, already presented in [10]. We transform (3.9) into differential equations for the functions $c(y)$ and $C(y)$, scaled limits of the sequences c_k and C_k under the change of variable (4.1). We obtain

$$c(y) = C(y) - rC''(y) \tag{4.2a}$$

$$C''(y) = u_0(y)C(y). \tag{4.2b}$$

Here and in the following, primes denote differentiations wrt y . The ‘potential’ function $u_0(y)$ reads

$$u_0(y) = \frac{1 - v(y)}{1 - rv(y)} \quad v(y) = \exp(-e^{\lambda y}). \tag{4.3}$$

In the low-temperature limit, we have $\lambda \rightarrow \infty$, and hence both $v(y)$ and $u_0(y)$ go to $\theta(y)$, Heaviside’s step function, except in a small region of size $1/\lambda$ around the origin. Thus the solution of (4.2) which goes to zero for $k \rightarrow \infty$, i.e. $y \rightarrow +\infty$, is such that

$$\begin{aligned} C(y) &= C_{(3)} e^{-y} & y > 0 \\ C(y) &= C_{(1)}y + C_{(2)} & y < 0 \end{aligned} \tag{4.4}$$

where $C_{(1)}$, $C_{(2)}$, and $C_{(3)}$ are unknown constants.

The determination of these numbers involves two steps, which will occur for every function in the following, namely

- (a) the analysis of the crossover region around the origin;
- (b) the matching at $y = y_0$ with the solution for finite values of k of the original difference equation.

4.1.1. *Function C , step (a).* The most systematic way of dealing with the vicinity of $y = 0$ is to use Laplace transforms. We define the Laplace transform $L_c(z)$ of the function $C(y)$ by ($0 < \text{Re } z < 1$)

$$L_c(z) = \int e^{-zy} C(y) dy \tag{4.5}$$

and the Laplace transform of the function $u_0(y)$ with a slightly different convention ($0 < \text{Re } z < \lambda$)

$$\mathcal{U}_0(z) = \int e^{-zy} u_0(y) dy. \tag{4.6}$$

An elementary calculation yields

$$\mathcal{U}_0(z) = \frac{1}{z} \Gamma\left(1 - \frac{z}{\lambda}\right) f_0\left(\frac{z}{\lambda}\right) \tag{4.7a}$$

where

$$f_0(s) = \frac{p}{r} \sum_{n \geq 1} r^n n^s \tag{4.7b}$$

is an analytic function in the whole complex plane. The differential equation (4.2b) is then equivalent to $(0 < \text{Re } s < \lambda, 0 < \text{Re}(z + s) < 1)$

$$z^2 L_c(z) = \int \frac{ds}{2\pi i} \mathcal{U}_0(s) L_c(z + s). \tag{4.8}$$

The connection with the constants $C_{(1)}$, $C_{(2)}$, and $C_{(3)}$ introduced above is simple. It follows from the definition of these numbers that $-C_{(3)}$ is the residue of $L_c(z)$ at $z = 1$, and that $L_c(z)$ has a double pole at $z = 0$, of the form $-C_{(1)}/z^2 + C_{(2)}/z$. It is advantageous to shift the s contour in (4.8) to the right of the pole at $s = 1 - z$, obtaining $(1 - \text{Re } z < \text{Re } s < \lambda)$

$$z^2 L_c(z) = C_{(3)} \mathcal{U}_0(1 - z) + \int \frac{ds}{2\pi i} \mathcal{U}_0(s) L_c(z + s). \tag{4.9}$$

This expression is well adapted to the analysis of the low-temperature limit, since the integral can be shown to be smaller by a factor of T^3 than the other terms. This property has already been used in [10]. Hence we have

$$L_c(z) = C_{(3)} \frac{\mathcal{U}_0(1 - z)}{z^2} + \mathcal{O}(T^3). \tag{4.10a}$$

This relation yields in particular

$$C_{(1)} = -C_{(3)} \mathcal{U}_0(1) \quad C_{(2)} = -C_{(3)} \mathcal{U}'_0(1) \tag{4.10b}$$

up to terms of order T^3 .

4.1.2. Function C , step (b). In order to perform this step of the analysis, we notice that the RHS of the difference equation (3.9b) is exponentially small in temperature, and hence negligible. The remaining equation is explicitly soluble, and yields

$$C_k = 1 + (C_2 - 1) \sum_{1 \leq l \leq k-1, l \text{ odd}} \frac{1}{l} \underset{k \rightarrow \infty}{\approx} 1 + \frac{C_2 - 1}{2} (\ln 2k + \gamma) \tag{4.11}$$

where γ denotes Euler's constant. The comparison of the large- k behaviour with (4.4) and (4.10b) yields

$$\begin{aligned} C_{(3)} &= \frac{-1}{(y_0 - \gamma/\lambda) \mathcal{U}_0(1) + \mathcal{U}'_0(1)} \\ &= \frac{1}{1 - y_0} + \frac{-(1 - y_0)\gamma + y_0 s_1}{(1 - y_0)^2 \lambda} \\ &\quad + \frac{(1 - y_0)^2 \gamma^2 - 2y_0(1 - y_0)\gamma s_1 + 2y_0^2 s_1^2 + (1 - y_0^2)(s_2 + \pi^2/6)}{2(1 - y_0)^3 \lambda^2} + \mathcal{O}(\lambda^{-3}) \end{aligned} \tag{4.12a}$$

$$\begin{aligned} C_{(1)} &= \frac{\lambda}{2} (C_2 - 1) \\ &= -C_{(3)} \mathcal{U}_0(1) = \frac{-1}{1 - y_0} - \frac{s_1}{(1 - y_0)^2 \lambda} - \frac{y_0 s_1^2 + (1 - y_0)(s_2 + \pi^2/6)}{(1 - y_0)^3 \lambda^2} + \mathcal{O}(\lambda^{-3}). \end{aligned} \tag{4.12b}$$

We have made use of the following expansions:

$$u_0(1) = 1 + \frac{\gamma + s_1}{\lambda} + \mathcal{O}(\lambda^{-2}) \quad u'_0(1) = -1 + \mathcal{O}(\lambda^{-2}). \quad (4.12c)$$

The s_k are defined by

$$s_k = f_0^{(k)}(0) = \frac{p}{r} \sum_{n \geq 1} r^n (\ln n)^k. \quad (4.13a)$$

Their small- p behaviour has been derived in [10]:

$$s_k \underset{p \rightarrow 0}{\approx} |\ln p|^k [1 - k\gamma/(\ln p)]. \quad (4.13b)$$

We have in particular

$$s_1 \underset{p \rightarrow 0}{\approx} -(\ln p + \gamma). \quad (4.13c)$$

By inserting (4.12b) into (3.11), we obtain the expansion $F = E_0 - S_0 T - \Gamma_0 T^2/2 + \dots$ of the free energy, already given in [10]. The ground-state energy E_0 reads

$$E_0 = -J - \frac{pH^2}{J+H}. \quad (4.14a)$$

The zero-temperature entropy S_0 is

$$S_0 = \frac{pH^2 s_1}{2(J+H)^2} \underset{p \rightarrow 0}{\sim} \frac{p|\ln p|H^2}{2(J+H)^2}. \quad (4.14b)$$

The specific heat $C(T)$ is linear in T at low temperature, with an amplitude

$$\Gamma_0 = \lim_{T \rightarrow 0} \frac{C(T)}{T} = \frac{pH^2}{2(J+H)^3} s_1^2 + \frac{pH}{2(J+H)^2} (s_2 - s_1^2 + \pi^2/6) \underset{p \rightarrow 0}{\sim} \frac{p(\ln p)^2 H^2}{2(J+H)^3}. \quad (4.14c)$$

4.2. The function D

The first part of the low-temperature analysis of the correlation function $\chi(q)$ deals with the complex quantities d_k and D_k , defined by (3.17a, b). It turns out that only the leading $T \rightarrow 0$ behaviour of these quantities is needed to evaluate the $\mathcal{O}(T)$ corrections to the correlation function. We first eliminate d_k between (3.18a, b), and transform the remaining equation for D_k into a differential equation for the unknown function $D(y)$, scaled limit of the quantities

$$\tilde{D}_k = kD_k \quad (4.15)$$

under the change of variable (4.1). We end up with

$$D''(y) = \frac{1 - v(y) e^{iq}}{1 - rv(y) e^{iq}} D(y) - \frac{2}{\lambda} \frac{v(y)}{1 - rv(y) e^{iq}} c'(y). \quad (4.16)$$

In the $T \rightarrow 0$ limit, we have $v(y) \rightarrow \theta(y)$, and (4.16) can be simplified, except in a small region around $y = 0$. For positive y , we have $D''(y) = D(y)$, and hence the decreasing solution is

$$D(y) = D_{(3)} e^{-y} \quad y > 0. \quad (4.17)$$

For negative y , we have

$$D''(y) = \omega^2 D(y) - \frac{2}{\lambda} \frac{C_{(1)}}{1 - r e^{iq}} \tag{4.18}$$

where ω^2 is a complex constant related to e^{iq} through

$$\omega^2 = \frac{1 - e^{iq}}{1 - r e^{iq}} \quad e^{iq} = \frac{1 - \omega^2}{1 - \omega^2 r}. \tag{4.19}$$

The solution of (4.18) is

$$D(y) = \frac{2}{\lambda} \frac{1 - \omega^2 r}{\omega^2 p} C_{(1)} + D_{(1)} e^{\omega y} + D_{(2)} e^{-\omega y} \quad y < 0 \tag{4.20}$$

where ω denotes the root of (4.19) with a positive real part.

Just as in the previous section, the determination of the constants $D_{(1)}$, $D_{(2)}$ and $D_{(3)}$ involves two steps.

4.2.1. Function D , step (a). We define the Laplace transform $L_D(z)$ of the function $D(y)$ by

$$L_D(z) = \int e^{zy} D(y) dy. \tag{4.21}$$

This definition is meaningful only for $\text{Re } \omega < \text{Re } z < 1$. The results which follow will be continued, if necessary, to the region $\text{Re } \omega \geq 1$ by analyticity in ω . We also define for further reference the function

$$u_q(y) = \frac{1 - v(y)}{1 - rv(y) e^{iq}} \tag{4.22}$$

and its Laplace transform, in analogy with (4.6a) ($0 < \text{Re } z < \lambda$)

$$\mathcal{U}_q(z) = \int e^{-zy} u_q(y) dy. \tag{4.23}$$

An elementary calculation yields

$$\mathcal{U}_q(z) = \frac{1}{z} \Gamma\left(1 - \frac{z}{\lambda}\right) f_q\left(\frac{z}{\lambda}\right) \tag{4.24a}$$

with

$$f_q(s) = \frac{p}{r(1 - \omega^2)} \sum_{n \geq 1} (r e^{iq})^n n^s. \tag{4.24b}$$

For $q = 0$, these definitions coincide with (4.7). The differential equation (4.16) is then equivalent to ($0 < \text{Re } s < \lambda$, $\text{Re } \omega < \text{Re}(z + s) < 1$)

$$(z^2 - \omega^2) L_D(z) = -\alpha z (1 - rz^2) L_c(z) + \int \frac{ds}{2\pi i} \mathcal{U}_q(s) [(1 - \omega^2) L_D(z + s) + \alpha (z + s) (1 - r(z + s)^2) L_c(z + s)] \tag{4.25}$$

with the notation

$$\alpha = \frac{2}{\lambda} \frac{\omega^2 r - 1}{p}. \tag{4.26}$$

In the derivation of (4.25), we have used the equality $\int e^{zy} c(y) dy = (1 - rz^2) L_c(z)$, which follows from (4.2a) and (4.5). Notice that the first term on the RHS of (4.25) is cancelled out if the contour of the integral is shifted to the left of $s = 0$. Instead of doing so, we prefer to shift the contour to the right of $s = 1 - z$, along the lines of the previous section. Taking into account that the residues of $L_c(z)$ and $L_D(z)$ at $z = 1$ are respectively $-C_{(3)}$ and $-D_{(3)}$, we obtain $(1 - \text{Re } z < \text{Re } s < \lambda)$

$$\begin{aligned} (z^2 - \omega^2)L_D(z) &= -\alpha z(1 - rz^2)L_C(z) + \mathcal{U}_q(1 - z)[(1 - \omega^2)D_{(3)} + \alpha pC_{(3)}] \\ &+ \int \frac{ds}{2\pi i} \mathcal{U}_q(s)[(1 - \omega^2)L_D(z + s) + \alpha(z + s)(1 - r(z + s)^2)L_C(z + s)]. \end{aligned} \tag{4.27}$$

For values of z such that $|z| \ll \lambda$, the leading behaviour as $T \rightarrow 0$ of the integral in (4.27) comes entirely from the term which contains L_C . By inserting the expression (4.10a) for $L_C(s)$, and performing the change of variable $s = \lambda t$ in the integral, we are left with

$$(z^2 - \omega^2)L_D(z) = -\alpha \frac{1 - rz^2}{z} C_{(3)} \mathcal{U}_0(1 - z) + \mathcal{U}_q(1 - z)[(1 - \omega^2)D_{(3)} + \alpha pC_{(3)}] + \alpha rC_{(3)} \mathcal{F}_D \tag{4.28}$$

with the definition

$$\mathcal{F}_D = \int \frac{dt}{2i \sin \pi t} f_q(t) f_0(-t) \quad 0 < \text{Re } t < 1. \tag{4.29}$$

This integral can be evaluated in closed form by using the expansions (4.7b) and (4.24b) of the functions f_0 and f_q , respectively, as well as the formula

$$\int \frac{dt}{2i \sin \pi t} x^t = \frac{x}{1 + x} \quad 0 < \text{Re } t < 1.$$

We obtain

$$\mathcal{F}_D = \frac{\omega^2 r - 1}{\omega^4 r^2} [\omega^2 r + \ln(1 - \omega^2 r)]. \tag{4.30}$$

The constants $D_{(1)}$ and $D_{(2)}$ are the residues of $L_D(z)$ at $z = -\omega$ and $z = \omega$, respectively. By inserting into (4.28) the value of $C_{(3)}$, given by (4.12a), we obtain, to leading order at low temperature

$$2\omega D_{(2)} - (1 + \omega)D_{(3)} = \frac{\alpha}{1 - y_0} \left(r \mathcal{F}_D + \frac{\omega^2 r - 1}{\omega(1 - \omega)} + \frac{p}{1 - \omega} \right) \tag{4.31}$$

together with a similar equation, where $D_{(1)}$ replaces $D_{(2)}$, and the sign of ω is reversed.

4.2.2. *Function D, step (b).* Going back to the difference equation (3.18b), we notice that the expression between the square brackets is smaller by a factor of T^2 than the quantities d_k or D_k themselves. In terms of the \tilde{D}_k introduced in (4.15), this is

$$(k + 1)(\tilde{D}_{k+2} - \tilde{D}_k) = (k - 1)(\tilde{D}_k - \tilde{D}_{k-2})[1 + \mathcal{O}(T^2)]. \tag{4.32}$$

This equation, which is still formally valid for $k = 1$, implies that the difference $\tilde{D}_k - \tilde{D}_{k-2}$ is also smaller than \tilde{D}_k by a factor of T^2 , and, going to the differential

formalism, that $D'(y_0)$ is smaller than $D(y_0)$ by a factor of T^2 . Setting $D'(y_0) = 0$ in (4.20) yields

$$D_{(1)} e^{\omega y_0} = D_{(2)} e^{-\omega y_0}. \tag{4.33}$$

This equation, together with (4.31), yields, again to leading order in temperature

$$D_{(1)} = \frac{A e^{-\omega y_0}}{\cosh \omega y_0} \quad D_{(2)} = \frac{A e^{\omega y_0}}{\cosh \omega y_0} \quad D_{(3)} = \frac{2}{\lambda(1-y_0)} \frac{\omega^2 r - 1}{\omega^2 p} + 2A \tag{4.34}$$

with the notation

$$A = -\frac{(1-\omega^2 r)^2 \ln(1-\omega^2 r)}{\lambda(1-y_0)\omega^4 p r(1-\omega \tanh \omega y_0)}. \tag{4.35}$$

The leading low-temperature behaviour of the quantity S_1 , defined in (3.18e), and entering the expression (3.20) of the correlation function, is easily derived from the above results:

$$S_1 = \frac{-2(1-\omega^2)(1-\omega^2 r)^2 \ln(1-\omega^2 r)}{\lambda(1-y_0) p r \omega^4 \cosh \omega y_0 (1-\omega \tanh \omega y_0)} + \text{cc}. \tag{4.36}$$

4.3. The function G .

We now perform the second part of the analysis of the correlation function, which deals with the real quantities g_k and G_k , defined by (3.13) and (3.17e). In analogy with the evaluation of d_k and D_k , we eliminate g_k between (3.18c, d), and derive a differential equation for the function $G(y)$, scaled limit of the quantities

$$\tilde{G}_k = k G_k \tag{4.37}$$

under the change of variable (4.1). For convenience, we also introduce the function $S(y)$, scaled limit of the quantity S_k defined in (3.18e). We end up with

$$G''(y) = u_0(y) G(y) - \frac{2}{\lambda} \frac{v(y)}{1-rv(y)} S'(y) \tag{4.38}$$

$$S(y) = \left[\frac{1}{1-rv(y)} e^{iq} \left(\frac{2}{\lambda} c'(y) + p e^{iq} D(y) \right) + \text{cc} \right] - \frac{2}{\lambda} c'(y) \tag{4.39a}$$

$$= [e^{iq}(D(y) - rD''(y)) + \text{cc}] + \frac{2}{\lambda} c'(y). \tag{4.39b}$$

These expressions can be simplified in the $T \rightarrow 0$ limit. For positive y , we obtain $G''(y) = G(y)$, and hence the decreasing solution is

$$G(y) = G_{(3)} e^{-y} \quad y > 0. \tag{4.40}$$

For negative y , we have

$$G''(y) = -\frac{2}{\lambda} \left(\frac{e^{iq}}{1-r e^{iq}} D'(y) + \text{cc} \right). \tag{4.41}$$

The expression (4.20) of the function D yields

$$G(y) = G_{(1)} y + G_{(2)} - \frac{2}{p\lambda} \left[\frac{1-\omega^2}{\omega} (D_{(1)} e^{\omega y} - D_{(2)} e^{-\omega y}) + \text{cc} \right]. \tag{4.42}$$

The determination of the constants $G_{(1)}$, $G_{(2)}$ and $G_{(3)}$ involves the usual two steps.

4.3.1. *Function G, step (a).* We define the Laplace transforms of the functions $G(y)$ and $S(y)$ by

$$L_G(z) = \int e^{zy} G(y) dy \quad L_S(z) = \int e^{zy} S(y) dy. \tag{4.43}$$

These definitions hold for $0 < \text{Re } z < 1$ and $\text{Re } \omega < \text{Re } z < 1$, respectively. Equation (4.38) is then equivalent to $(0 < \text{Re } s < \lambda, \text{Re } \omega < \text{Re}(z + s) < 1)$

$$z^2 L_G(z) = \int \frac{ds}{2\pi i} \mathcal{U}_0(s) L_G(z+s) + \frac{2}{\lambda p} \left(z L_S(z) - \int \frac{ds}{2\pi i} \mathcal{U}_0(s) (z+s) L_S(z+s) \right) \tag{4.44}$$

and the Laplace transformation of (4.39) yields

$$L_S(z) = -\frac{2}{\lambda} z(1-rz^2) L_C(z) + (1-rz^2)(e^{iq} L_D(z) + cC). \tag{4.45}$$

The function L_D is given by (4.27), where the integral of $L_D(z+s)$, being of relative order T^2 , can be omitted. By shifting the s contour in (4.44) to the right of $s = 1 - z$, we obtain finally $(1 - \text{Re } z < \text{Re } s < \lambda)$

$$z^2 L_G(z) = \int \frac{ds}{2\pi i} \mathcal{U}_0(s) L_G(z+s) + \frac{2}{\lambda p} \left(z L_S(z) - \int \frac{ds}{2\pi i} \mathcal{U}_0(s) (z+s) L_S(z+s) \right) + \mathcal{U}_0(1-z) \left(G_{(3)} + \frac{4}{\lambda^2 p} C_{(3)} - \frac{2}{\lambda} (e^{iq} D_{(3)} + cC) \right). \tag{4.46}$$

The unknown constants $G_{(1)}$ and $G_{(2)}$ are such that the divergent part of the function $L_G(z)$ around $z=0$ reads $-G_{(1)}/z^2 + G_{(2)}/z$. The first two terms of the T expansion of these numbers can be extracted from (4.46) through a lengthy but straightforward calculation. The unknown constant $G_{(3)}$ will be determined in step (b). The central part of the computation is the expansion of the integral

$$\mathcal{I}_G = - \int \frac{ds}{2\pi i} \mathcal{U}_0(s) (z+s) L_S(z+s) = \mathcal{I}_1 + \frac{1}{\lambda} (\mathcal{I}_2 + \mathcal{I}_3 z) + \dots \tag{4.47}$$

The leading term \mathcal{I}_1 is the following combination of integrals:

$$\mathcal{I}_1 = \frac{2H}{J+H} \left[\left(\frac{r^2}{p} (\omega^2 - 1)(K - I) + cC \right) + rI \right] \tag{4.48}$$

with $(0 < \text{Re } t < 1, 0 < \text{Re } u < 1)$

$$I = \int \frac{dt}{2i \sin \pi t} t f_0(t) f_0(-t)$$

$$K = \int \int \frac{dt}{2\pi i} \frac{du}{2\pi i} \frac{1}{u} \Gamma(1-t) f_0(t) \Gamma(1-u) f_q(u) \Gamma(1+t+u) f_0(-t-u). \tag{4.49}$$

These integrals can be evaluated using the expansions of the functions f_0 and f_q , in analogy with the derivation of (4.30). Our final expression is

$$\mathcal{I}_1 = \frac{H}{J+H} \chi_{(\infty)}(q) \tag{4.50}$$

where

$$\chi_{(\infty)}(q) = \sum_{-\infty < l < +\infty} \chi_{(\infty),l} e^{iq}$$

with

$$\chi_{(\infty),l} = \frac{1}{3} p^2 r^{l-1} \sum_{N \geq 2} \frac{N(N^2-1)r^N}{(N+l)^2} \quad l \geq 0 \tag{4.51}$$

is just the $H = \infty$ limit of the correlation function, derived in the appendix, and given in closed form by (A10). This quantity appears here in a very different way. Notice that $2rI = \chi_{(\infty),0}$ is nothing but the $l = 0$ term of (4.51), and that I is also the contribution that would come from the pole at $u = 0$ in the integral K , had we shifted the u contour to the left of that pole.

The calculation of the correction terms $\mathcal{F}_2, \mathcal{F}_3$ is more lengthy. It turns out that only the difference $\mathcal{F}_2 - \mathcal{F}_3$ will be needed in the following, and that this difference is simpler to evaluate than each term separately. We prefer not to bother the reader with the building blocks of the derivation: they are complex integrals which resemble those of (4.49). We obtain

$$\mathcal{F}_2 - \mathcal{F}_3 = \frac{H}{J+H} \left[\left(\gamma - 2 + \frac{Js_1}{J+H} \right) \chi_{(\infty)}(q) + 2\chi_{(L)}(q) + \Psi(q) \right] \tag{4.52}$$

where s_1 is defined in (4.13a). This quantity appears through the $\mathcal{O}(T)$ correction to $C_{(3)}$, given by (4.12a).

The expression of $\chi_{(L)}(q)$ differs from that of $\chi_{(\infty)}(q)$ only by a logarithmic factor

$$\begin{aligned} \chi_{(L)}(q) &= \sum_{-\infty < l < \infty} \chi_{(L),l} e^{iq} \\ \chi_{(L),l} &= \frac{1}{3} p^2 r^{l-1} \sum_{N \geq 2} \frac{N(N^2-1)r^N \ln(N+l)}{(N+l)^2} \quad l \geq 0. \end{aligned} \tag{4.53}$$

The function $\Psi(q)$ can be expressed in a closed form as

$$\begin{aligned} \Psi(q) &= \frac{2}{pr^2\omega^8} (1-\omega^2)(1-\omega^2r) \left[\omega^2r(\omega^2r-2)\Lambda - 2\omega^4r^2 + [\omega^2r + (1-\omega^2r)\Lambda] \right. \\ &\quad \left. \times \left(\omega^2r + (1-\omega^2) \frac{\Lambda}{1-\omega \tanh \omega y_0} \right) \right] + \text{cc} \end{aligned} \tag{4.54}$$

with the notation

$$\Lambda = \ln(1-\omega^2r). \tag{4.55}$$

The first two terms in the square brackets in (4.54) come from the function L_C in (4.27), while the terms in the larger parentheses come from the amplitudes $D_{(3)}$ and $C_{(3)}$.

The expansion of the other terms of (4.46) is easier. We finally obtain

$$G_{(1)} = -G_{(3)} \mathcal{U}_0(1) + \Delta_{(1)} \tag{4.56a}$$

$$\Delta_{(1)} = -\frac{2}{p\lambda} \mathcal{F}_1 - \frac{2}{p\lambda^2} \mathcal{F}_2 + \frac{2}{\lambda} \left[e^{iq} \left(D_{(3)} - \frac{1}{p} D_{(0)} \right) + \text{cc} \right] \tag{4.56b}$$

$$G_{(2)} = -G_{(3)} \mathcal{U}'_0(1) + \Delta_{(2)} \tag{4.56c}$$

$$\Delta_{(2)} = \frac{2}{p\lambda^2} \mathcal{F}_3 - \frac{2}{\lambda} \left[e^{iq} \left(D_{(3)} - \frac{1}{p} D_{(F)} \right) + \text{cc} \right]. \tag{4.56d}$$

These expression involve the residue $D_{(0)}$, and the finite part $D_{(F)}$, of the function $L_D(z)$ at $z = 0$: $L_D(z) = D_{(0)}/z + D_{(F)} + \mathcal{O}(z)$. It is sufficient to know both of these

quantities to leading order in temperature. $D_{(0)}$ is just the constant term in the RHS of (4.20); $D_{(F)}$ is easily evaluated from (4.34):

$$D_{(F)} = D_{(0)} + 2A \left(1 - \frac{\tanh \omega y_0}{\omega} \right). \tag{4.57}$$

4.3.2. *Function G , step (b).* In the difference equation (3.18d), the expression between the square brackets is smaller by a factor of T^2 than the other terms, such as the quantities g_k or G_k themselves. Since we are only interested in the first two terms of the T expansion of the G_k , it is therefore legitimate to set the above-mentioned expression to zero. In terms of the \tilde{G}_k defined by (4.37), this condition is

$$(k+1)(\tilde{G}_{k+2} - \tilde{G}_k) = (k-1)(\tilde{G}_k - \tilde{G}_{k-2}). \tag{4.58}$$

Hence the quantity $(k-1)(\tilde{G}_k - \tilde{G}_{k-2})$ is a constant, equal to $\tilde{G}_2 = 2G_2$. We have therefore

$$\tilde{G}_k = 2G_2 \sum_{1 \leq l \leq k-1, \text{odd}} \frac{1}{l} \underset{k \rightarrow \infty}{\approx} G_2 (\ln 2k + \gamma). \tag{4.59}$$

The final step consists in matching this result with (4.42), using (4.33). This leads to

$$G(y_0) = G_2 \gamma = G_{(1)y_0} + G_{(2)} \tag{4.60a}$$

$$G'(y_0) = G_2 \lambda = G_{(1)} + K \tag{4.60b}$$

with

$$K = -\frac{4}{\lambda p} \left(\frac{(1-\omega^2)A}{\cosh \omega y_0} + c.c. \right). \tag{4.60c}$$

Equations (4.56) and (4.60) finally yield

$$G_2 = -\frac{C_{(3)}}{\lambda} \{ K[y_0 \mathcal{U}_0(1) + \mathcal{U}'_0(1)] + \Delta_{(1)} \mathcal{U}'_0(1) - \Delta_{(2)} \mathcal{U}_0(1) \}. \tag{4.61}$$

The calculation of the correlation function $\chi(q)$ is now very simple. We insert (4.36) and (4.61) into (3.20), and use the expansion (4.12) of the quantities $C_{(3)}$, $\mathcal{U}_0(1)$ and $\mathcal{U}'_0(1)$, to first order in $1/\lambda$. The term proportional to K in (4.61) cancels the contribution proportional to S_1 . Moreover, all the contributions proportional to Euler's constant γ also cancel out. The zero-temperature result only originates in the integral \mathcal{I}_1 . In the correction linear in T , there is also a cancellation of contributions linear in Λ , defined in (4.55). The final results are presented in § 5.

5. Low-temperature behaviour: results

5.1. The correlation function

In the previous section, we have performed the low-temperature analysis of the exact expression, derived in § 3, for the correlation function $\chi(q)$. The final expression for this correlation function at low temperature, obtained as explained at the end of § 4, has the form

$$\chi(q) = \chi^{(0)}(q) + T\chi^{(1)}(q) + \mathcal{O}(T^2). \tag{5.1}$$

The zero-temperature value is simply proportional to the correlation function $\chi_{(\infty)}$ of the $H \rightarrow \infty$ limit of the model

$$\chi^{(0)}(q) = \left(\frac{H}{J+H}\right)^2 \chi_{(\infty)}(q). \tag{5.2}$$

The function $\chi_{(\infty)}$, given by (4.51), is derived in the appendix, along the lines of [4], and given explicitly in (A10). The simple H dependence in (5.2) is reminiscent of the results (4.14) for thermodynamic properties.

The correction $\chi^{(1)}$ linear in T has a much more complex and interesting structure. It contains two very different types of terms, namely

$$\chi^{(1)}(q) = \chi^{(1.1)}(q) + \chi^{(1.2)}(q). \tag{5.3}$$

The first contribution contains only the function $\chi_{(\infty)}$ of the $H = \infty$ limit, and the very similar function $\chi_{(L)}$, defined in (4.53)

$$\chi^{(1.1)} = \frac{H}{(J+H)^2} (\chi_{(L)} - \chi_{(\infty)}) - \frac{JH}{(J+H)^3} s_1 \chi_{(\infty)} \tag{5.4}$$

where s_1 has been defined in (4.13a). Thus $\chi^{(0)}$ and $\chi^{(1.1)}$ have an explicit rational H dependence, whereas their p dependence cannot be expressed in terms of elementary functions.

The second contribution to (5.3) is obtained in closed form as

$$\chi^{(1.2)} = \frac{H}{(J+H)^2} \frac{(1-\omega^2)(1-\omega^2 r)}{pr^2 \omega^8} \Sigma + c.c. \tag{5.5a}$$

with

$$\Sigma = \frac{(1-\omega^2)(1-\omega^2 r)}{1-\omega \tanh \omega y_0} \ln^2(1-\omega^2 r) - \omega^4 r^2. \tag{5.5b}$$

We recall that $y_0 = -J/H$, and that the complex constant ω has been defined in (4.19). Hence $\chi^{(1.2)}$ is also given by

$$\chi^{(1.2)} = \frac{pH}{(J+H)^2} \left[\frac{1}{2 \sin^2 q/2} + \frac{p^2}{r^2 (2 \sin q/2)^4} \left(\frac{\ln^2[p/(1-r e^{iq})]}{1 + \omega \tanh(\omega J/H)} + c.c. \right) \right]. \tag{5.5c}$$

The dependence of this expression on both p and q is partly coded in the complex parameter ω .

We now analyse some particular cases and present some illustrations of the central results of the paper, expressed in (5.1-5).

5.2. The correlation length

It follows from the expression (5.1-5) of the correlation function that the correlation length ξ is discontinuous at zero temperature.

Indeed, the value of $\chi(q)$ at $T = 0$ given in (5.2) yields the correlation length of the $H = \infty$ limit at zero temperature. Namely, with the notation (3.24)

$$\zeta = e^{-1/\xi} = r \quad T = 0, \text{ any } H. \tag{5.6}$$

On the other hand, the term $\chi^{(1.2)}(q)$, linear in temperature, given by (5.5), is clearly singular for

$$\omega \tanh(\omega J/H) = -1. \tag{5.7}$$

Equation (5.7) has a purely imaginary solution $\omega = it$, where the real quantity t is implicitly given by

$$\frac{H}{J} = \frac{t}{\tan^{-1}(1/t)}. \tag{5.8}$$

This value of ω corresponds to a real value of e^{iq} lying between 0 and 1, and hence yields the limit of the correlation length as temperature goes to zero. With the notation of (3.24), we have

$$\zeta = \frac{1 + rt^2}{1 + t^2} \quad T \rightarrow 0. \tag{5.9}$$

This quantity depends on H in a monotonic way, between the bounds $r \leq \zeta \leq 1$. Hence the limit of the correlation length as $T \rightarrow 0$ is strictly larger than the $T = 0$ value, for any finite value of H . For large H , we have $t^2 \sim H/J$, so that the $H = \infty$ value (5.6) has a regular correction

$$\zeta = r + \frac{pJ}{H} + \dots \quad T \rightarrow 0, H \rightarrow \infty. \tag{5.10}$$

For small H , we have $t \sim \pi H/(2J)$, and therefore the correlation length diverges as

$$\xi \sim \frac{4J^2}{\pi^2 p H^2} \quad T \rightarrow 0, H \rightarrow 0. \tag{5.11}$$

This behaviour has a simple explanation: $1/\xi$ is proportional to the strength of disorder $\langle h_n^2 \rangle = 2 p H^2$.

For small p , the correlation length scales as $1/p$

$$\Xi = \lim_{p \rightarrow 0} p \xi = \frac{t^2 + 1}{t^2}. \tag{5.12}$$

The amplitude Ξ has a non-trivial H dependence: t and H are related by (5.8). The more general analysis of the scaling behaviour of $\chi(q)$ for small values of p and q will be given in § 5.6.

Figure 1 shows a plot of the quantity $\zeta = e^{-1/\xi}$, against temperature, for $p = 0.7$, and different values of H/J . The data have been obtained by iterating numerically (3.27), along the lines of § 3.3. The analytic $T \rightarrow 0$ limit (5.9) is very well reproduced. The arrow indicates the zero-temperature value (5.6), which is independent of H , and hence the same for all curves.

5.3. The susceptibility

The usually defined susceptibility χ is just the value at zero wavevector $\chi(0)$ of the correlation function.

The susceptibility in the $H = \infty$ limit is given in the appendix:

$$\chi_{(\infty)}(0) = \frac{2 - p - p^2}{6p}. \tag{5.13}$$

The value at $q = 0$ of the related quantity $\chi_{(L)}$ can be deduced from (4.53):

$$\chi_{(L)}(0) = \frac{p^2}{6r} \sum_{n \geq 2} r^n (n^2 - 1) \ln n \underset{p \rightarrow 0}{\sim} \frac{1}{3p} (-\ln p + \frac{3}{2} - \gamma). \tag{5.14}$$

As far as the quantity $\chi^{(1,2)}$ is concerned, the expansion of (5.5c) around $q = 0$, i.e.

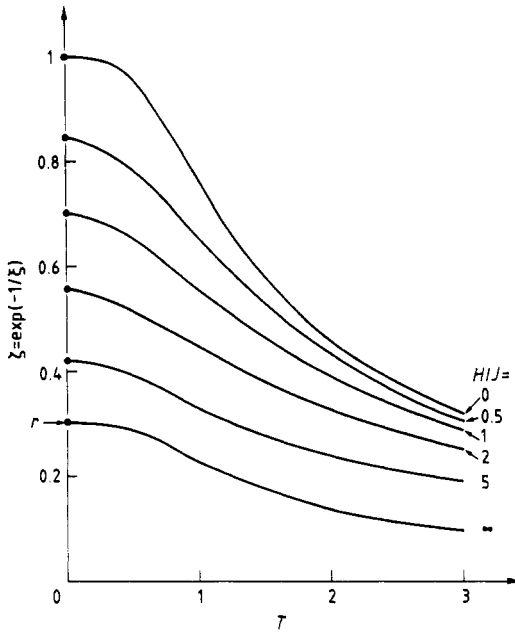


Figure 1. Plot of $\zeta = \exp(-1/\xi)$ against temperature, where ξ is the correlation length, for a dilution probability of $p = 0.7$, and different values of H , indicated on the curves. The limit values for $T \rightarrow 0$, marked by dots, are given by (5.9). At $T = 0$ strictly, we have $\zeta = r = 0.3$, independently of H ; this value is indicated by an arrow.

$\omega^2 = 0$, yields

$$\chi^{(1,2)}(0) = \frac{1}{3pH^2(J+H)^2} \left[2J^3 + 6J^2H + (3r+9)JH^2 + \left(-\frac{r^2}{2} + 3r + 3 \right) H^3 \right]. \quad (5.15)$$

The final expression of the susceptibility is simpler for small p , where it assumes the scaling form

$$\lim_{p \rightarrow 0} p\chi = \frac{H^2}{3(J+H)^2} \left(1 - (\ln p + \gamma) \frac{T}{J+H} \right) + \frac{2(J^3 + 3J^2H + 6JH^2 + 3H^3)T}{3H^2(J+H)^2} + \mathcal{O}(T^2). \quad (5.16)$$

A factor of $\ln p$ shows up among the corrections linear in temperature. Logarithms of p are already present in the terms of the low-temperature expansion of the free energy, such as the zero-temperature entropy, and the specific heat amplitude, given in (4.14b, c), and will occur in the scaling form of $\chi(q)$ for small values of p and q , presented in § 5.6.

Figure 2 shows a plot of the susceptibility against temperature, for different values of H/J . We have chosen the value $p = 1$, corresponding to no dilution. In this limiting case, the correlation function vanishes in the $T \rightarrow 0$ limit. The straight lines on the graph indicate the exact values of the slopes, extracted from (5.13-15).

5.4. The Edwards-Anderson parameter

We have recalled in § 1 (see equation (1.2)) how the Edwards-Anderson order parameter q_{EA} is related to the correlation function χ_0 at coinciding points. It is clear from

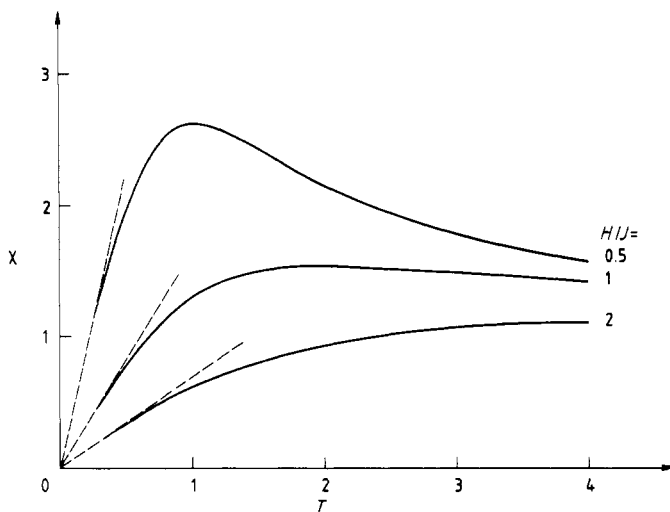


Figure 2. Plot of the usual ($q=0$) susceptibility χ , against temperature, for $p=1$, and different values of H , indicated on the curves. The susceptibility vanishes at $T=0$, and behaves linearly at low temperature; the slopes of the broken straight lines have been extracted from the exact result (5.13-15).

(1.1a) that this quantity can be determined from the above exact expression of the function $\chi(q)$ by forgetting about all terms proportional either to e^{iq} or to its complex conjugate in the difference equations derived in § 3. This amounts to setting $e^{iq} = 0$, i.e. $\omega^2 = 1$, in the low-temperature analysis, up to § 4.4.

We obtain

$$\chi_0 = \chi_0^{(0)} + T\chi_0^{(1)} + \mathcal{O}(T^2). \tag{5.17}$$

The zero-temperature value is

$$\chi_0^{(0)} = \left(\frac{H}{J+H}\right)^2 \chi_{(\infty),0} \tag{5.18a}$$

with

$$\chi_{(\infty),0} = \frac{1}{3} \left(1 + \frac{p^2 \ln p}{r}\right). \tag{5.18b}$$

The value at $H = \infty$ has been taken from (A9) of the appendix.

The correction linear in T is given by

$$\chi_0^{(1)} = \frac{1}{J+H} + \frac{H}{(J+H)^2} [\chi_{(L),0} - \chi_{(\infty),0}] - \frac{JH}{(J+H)^3} s_1 \chi_{(\infty),0}. \tag{5.19}$$

The contribution of $\chi_{(L)}$ can be derived from (4.53):

$$\chi_{(L),0} = \frac{p^2}{3r} \sum_{n \geq 1} \frac{(n^2 - 1)r^n \ln n}{n} \underset{p \rightarrow 0}{\sim} \frac{1}{3} (-\ln p + 1 - \gamma). \tag{5.20}$$

The final expression is again simpler for small p , where we have

$$\chi_0 = 1 - q_{EA} = \frac{H^2}{3(J+H)^2} \left(1 - (\ln p + \gamma) \frac{T}{J+H}\right) + \frac{T}{J+H} + \mathcal{O}(T^2). \tag{5.21}$$

Figure 3 shows a plot of χ_0 against temperature, for $p = 0.25$, and different values of H/J . The numerical data, obtained by solving the recursion relations from § 3, are in good agreement, up to $T/J \approx 0.4$, with the exact $T \rightarrow 0$ behaviour (5.17-20), indicated by straight lines.

5.5. *The $p = 1$ case*

The low-temperature expansion (5.1-5) of the correlation function simplifies drastically in the limiting case where $p = 1$, corresponding to no dilution: every spin is subjected to a non-zero random field. Indeed, if $p = 1$ and $H = \infty$ simultaneously, there is an infinite field at each site, and no correlation at all in the system, at any temperature. Hence both functions $\chi_{(\infty)}$ and $\chi_{(L)}$ vanish identically.

We thus obtain the *a priori* striking result that the correlation function $\chi(q)$ vanishes at zero temperature, for all values of H , when $p = 1$. In other words, the $T = 0$ correlations are entirely due to dilution, i.e. to the sites with no random field. This behaviour is analogous to that of the zero-temperature entropy S_0 , given in (4.14b), which also vanishes for $p = 1$.

For any finite value of H , the low-temperature behaviour of the correlation function comes entirely from the term $\chi^{(1,2)}$. The expansion of (5.5) around its apparent singularity at $r = 0$ easily yields

$$\chi(q) = \frac{H}{(J+H)^2} \left(\frac{1-\omega^2}{\omega^3} \frac{\tanh \omega y_0 - \omega}{1-\omega \tanh \omega y_0} + c.c. \right) T + \mathcal{O}(T^2) \tag{5.22}$$

and the definition (4.19) of ω^2 assumes the simpler form

$$\omega^2 = 1 - e^{iq}. \tag{5.23}$$

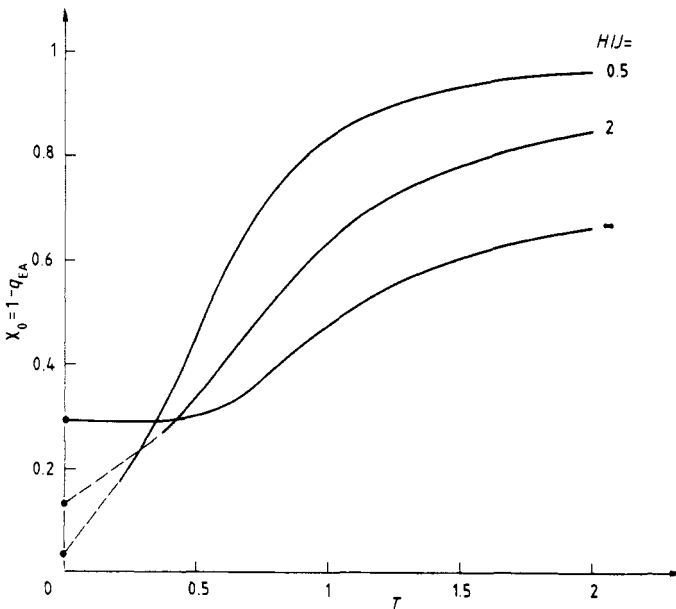


Figure 3. Plot of $\chi_0 = 1 - q_{EA}$ against temperature, where q_{EA} is the Edwards-Anderson order parameter, for $p = 0.25$, and different values of H , indicated on the curves. Both the zero-temperature values, and the first correction linear in temperature (broken lines), are given in (5.17-20).

5.6. Scaling behaviour for small p

If the fraction p of spins which experience a random field is small, the characteristic length between two such spins will be of the order of $1/p \gg 1$. Hence the correlation function can be expected to exhibit a scaling behaviour if the wavevector q is scaled proportionally to p . Indeed, in terms of the scaled variable Q such that

$$q = pQ \quad p \rightarrow 0, Q \text{ fixed} \tag{5.24}$$

the correlation function given by (5.1-5) takes the form

$$\lim_{p \rightarrow 0} p\chi(pQ) = \left(\frac{H}{J+H}\right)^2 \left[1 - (\ln p + \gamma) \frac{T}{J+H}\right] X(Q) + \frac{HT}{(J+H)^2} Y(Q) + \mathcal{O}(T^2) \tag{5.25}$$

with the definitions

$$X(Q) = \frac{2}{Q^4} [Q^2 + \ln(1 + Q^2) - 2Q \tan^{-1} Q] \tag{5.26a}$$

$$Y(Q) = \frac{2}{Q^4} \left[-\frac{1}{4} \ln^2(1 + Q^2) + Q \tan^{-1} Q \ln(1 + Q^2) + (\tan^{-1} Q)^2 + \frac{1}{2} \left(\frac{\ln^2(1 - iQ)}{1 + \omega \tanh(\omega J/H)} + \text{cc} \right) \right] \tag{5.26b}$$

and the scaling form of the complex constant ω reads

$$\omega^2 = \frac{Q}{Q + i}. \tag{5.27}$$

As mentioned in the appendix, the expression (5.26a) for the function $X(Q)$ is obtained by replacing in the definition (4.51) or (A8) the sum by an integral. An analogous manipulation on (4.53) yields the expression of $Y(Q)$.

In (5.25), one part of the $\mathcal{O}(T)$ corrections renormalises the zero-temperature result, whereas the other one involves the more intricate function Y , which admits the correlation length of (5.12). The functions $X(Q)$ and $Y(Q)$ have by far a more complicated structure than a simple Lorentzian or Lorentzian squared. For instance, their behaviour for a small scaled wavevector Q is

$$X(Q) = \frac{1}{3} - \frac{2}{15} Q^2 + \mathcal{O}(Q^4) \tag{5.28a}$$

$$Y(Q) = \frac{2(J^3 + 3J^2H + 6JH^2 + 3H^3)}{3H^3} + \mathcal{O}(Q^2) \tag{5.28b}$$

whereas we have, for large values of Q

$$X(Q) = \frac{2}{Q^2} - \frac{2\pi}{|Q|^3} + \frac{4(\ln|Q| + 1)}{Q^4} + \dots \tag{5.29a}$$

$$Y(Q) = \frac{2\pi \ln|Q|}{|Q|^3} + \frac{2}{Q^4} \left[-\ln^2|Q| - 2 \ln|Q| + \frac{\ln^2|Q| - \pi^2/4}{1 + \tanh(J/H)} + \pi^2/4 \right] + \dots \tag{5.29b}$$

The values at $Q = 0$ of the scaling functions X and Y clearly agree with the expression (5.16) of the small- p behaviour of the usual ($q = 0$) susceptibility.

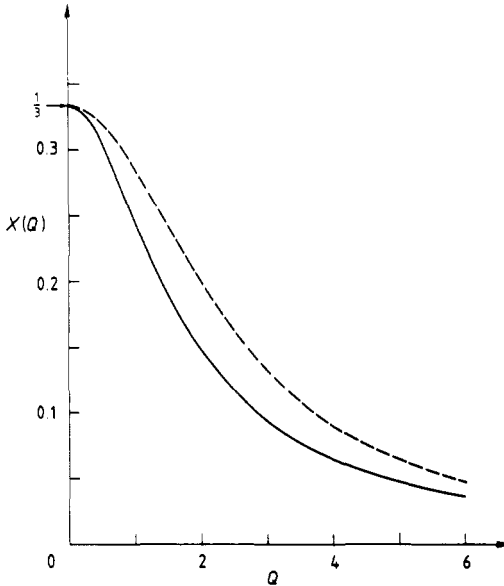


Figure 4. Plot of scaling form $X(Q)$ of the zero-temperature correlation function, defined in (5.26a) and (A8), against the scaled wavevector Q . The full curve shows $X(Q)$ itself; the broken curve shows the 'Lorentzian fit' defined in the text.

Figure 4 shows a plot of the leading scaling function $X(Q)$ (full curve), against the scaled wavevector Q , together with a 'Lorentzian fit' (broken curve). The Lorentzian curve, given by $\mathcal{X}(Q) = 2/(Q^2 + 6)$, fits the exact function in the sense that it has both the same value $\frac{1}{3}$ at $Q = 0$, and the same leading decay $2/Q^2$ for $Q \rightarrow \infty$. We notice that its correlation length is $1/\sqrt{6}$, instead of 1, and that the coefficient of Q^2 in its small- Q expansion is, in absolute value, $\frac{1}{18}$, instead of $\frac{2}{15}$. The form of the other scaling function $Y(Q)$ is very similar to that of $X(Q)$. If plotted against the variable $Q\Xi$, where Ξ is the correlation length amplitude defined in (5.12), $Y(Q)$ only exhibits a very weak H -dependence, in spite of its rather intricate analytical expression, involving the complex quantity ω .

6. Discussion

In the present paper, we have derived the exact expression of the correlation function, or wavevector-dependent susceptibility, $\chi(q)$, of the one-dimensional random-field Ising model, for a particular symmetric distribution of the magnetic fields. Its symmetric exponential form (1.3) mimics the diluted binary distribution naturally present e.g. in diluted antiferromagnets. Since the model is one dimensional, it is clearly in its paramagnetic phase at any finite temperature. One of our motivations was to explore the analytic structure of the correlation function $\chi(q)$, in order to see whether an exact calculation yields, at least in some limit, a Lorentzian form. Indeed, it has been shown [15] that the celebrated ' $d \rightarrow d - 2$ ' mapping from a random-spin system in dimension d onto the corresponding pure model in dimension $d - 2$ predicts that, in three dimensions, the full two-point function is a superposition of a Lorentzian and a Lorentzian squared: $S(q) \approx A(q^2 + \xi^{-2})^{-2} + B(q^2 + \xi^{-2})^{-1}$. The connected two-point

function, or susceptibility, studied in the present work, is rather expected to be a simple Lorentzian: $\chi(q) \sim C(q^2 + \xi^{-2})^{-1}$. Hence these simple functional forms for $S(q)$ and $\chi(q)$ are usually used to fit experimental data on random magnetic systems. A recent exact study [9] of a lattice gas model has also shown that the correlation function $S(q)$ of a random system can be a pure Lorentzian squared at zero temperature, even in one dimension. In higher dimensions, various approaches, like e.g. linear response theory, also lead to such simple forms. See [16] for a recent review of the subject.

The interest of the exact solution at finite temperature, described in § 3, is twofold. Indeed, it provides an efficient algorithm for obtaining numerical values, at finite (moderate or high) temperature, of the correlation function, and related quantities, such as the correlation length, or the Edwards–Anderson order parameter. On the other hand, we have been able to extract from the exact solution the low-temperature behaviour of the various physical quantities of interest. The technical part of the derivation is presented in § 4, whereas results are discussed in § 5. The leading corrections to the zero-temperature values are evaluated. They are linear in temperature, just as for thermodynamic quantities, studied previously by the authors in [10]. This smooth dependence is related to the absence of a gap in the excitation spectrum, which may be due to the continuous nature of the random field distribution. To our knowledge, the question whether there is a gap in the case of (diluted) binary disorder is still open.

The main features of the results are the following. First of all, we do not find any simple form for the correlation function, neither at finite temperature, nor at zero temperature, even in its scaling form in the $p \rightarrow 0$ limit of a very diluted system.

It is instructive to consider first the $H = \infty$ limit of our model, already studied in [4]. In this limiting case, any non-zero random field pins the spin on which it acts, thus breaking the chain into an infinity of independent finite systems. This is why the problem can be solved by direct enumeration. This method has been used in [4] to calculate both correlation functions $S(q)$ and $\chi(q)$. The results concerning the second function are recalled in the appendix. At any finite temperature, $\chi(q)$ is an infinite sum of lattice Lorentzians. For large distance, the correlation decays as a simple exponential, since it is obviously dominated by the first of these Lorentzians, i.e. that with the smallest ‘width’, and more generally by the complex singularity with the smallest imaginary part. However, the infinite series does contribute to the structure of the correlation function. In other words, for any fixed (real) value of the wavevector q , the correlation function is mostly determined by the behaviour of the correlations at finite distance in real space, and not only by their asymptotic decay. At zero temperature, the infinite series of poles merges into a logarithmic branch cut, and the exponential decay of the correlation is multiplied by the squared inverse of distance, as stated in (A13). This phenomenon, which has been overlooked in [4], is responsible for the non-triviality of the zero-temperature correlation function, even in the scaling regime at $p \rightarrow 0$. The full two-point function $S(q)$ exhibits the very same kind of behaviour. A careful analysis of the expressions given in [4] indeed shows that the zero-temperature limit of $S(q)$ also involves a logarithmic branch cut. Hence, even in the small- p scaling regime, its structure is again more complicated than the sum of a Lorentzian and a Lorentzian squared, in contradiction with the claims of [4].

For finite values of the strength H of the random fields, the zero-temperature value of the correlation function is simply proportional to that of the $H = \infty$ limit, and has therefore the same structure. The behaviour at small non-zero temperature is very different. Indeed, the correlation function has corrections linear in T , rather than proportional to $\exp(-2\beta J)$. These linear terms vanish in the $H \rightarrow \infty$ limit. Moreover,

their behaviour in the limit of a small probability involves powers of $|\ln p|$, just as thermodynamic quantities. The correlation length is discontinuous at zero temperature. Its $T=0$ value is independent of H , and therefore equal to the value for $H = \infty$. Its $T \rightarrow 0$ limit has a strong and non-trivial H -dependence; it is larger than the $T = 0$ value, because spins which are subjected to finite random fields do contribute to correlations at any finite temperature.

We end up by mentioning some related questions, which the present paper leaves open. If a uniform external field is superimposed on the symmetric random fields, the model is in principle still exactly soluble, although quite some complications would enter the present analysis, already rather heavy. It would also be desirable to find an efficient way of evaluating the full two-point function $S(q)$, with or without an external magnetic field.

Acknowledgments

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Appendix. The $H = \infty$ limit

This appendix is devoted to a detailed study of the $H = \infty$ limit of our exactly soluble model. As we have mentioned in § 1, the distribution of the random fields becomes very simple in that limit:

$$h_n = \begin{cases} -\infty & \text{with probability } p/2 \\ 0 & \text{with probability } r \\ +\infty & \text{with probability } p/2. \end{cases} \tag{A1}$$

This limit case has been studied, using a direct enumeration approach, by Grinstein and Mukamel, in [4]. For sake of consistency with the text, the correlation function of the $H = \infty$ limit is denoted in this appendix by $\chi_{(\infty),l}$, and its Fourier transform by $\chi_{(\infty)}(q)$. The central result of [4] is that $\chi_{(\infty),l}$ can be expressed in terms of one- and two-spin correlations of the pure Ising model, evaluated in finite systems, of length N , with appropriate boundary conditions

$$\chi_{(\infty),l} = \frac{1}{2} \sum_{N \geq 0} p^2 r^{N+l-1} \sum_{n \geq 0} [\langle \sigma_n \sigma_{n+l} \rangle_{N+l}^+ + \langle \sigma_n \sigma_{n+l} \rangle_{N+l}^- - \langle \sigma_n \rangle_{N+l}^+ \langle \sigma_{n+l} \rangle_{N+l}^+ - \langle \sigma_n \rangle_{N+l}^- \langle \sigma_{n+l} \rangle_{N+l}^-]. \tag{A2}$$

The explicit values of the correlations

$$\langle \sigma_n \rangle_N^\pm = \frac{\tau^n \pm \tau^{N-n}}{1 \pm \tau^N} \quad \langle \sigma_n \sigma_{n+l} \rangle_N^\pm = \frac{\tau^l \pm \tau^{N-l}}{1 \pm \tau^N} \tag{A3}$$

yield after some manipulations

$$\chi_{(\infty),l} = p^2 r^{l-1} \tau^l \sum_{N \geq 0} r^N \frac{1 + \tau^{2N+2l}}{(1 - \tau^{2N+2l})^2} \sum_{n=0}^N (1 - \tau^{2n})(1 - \tau^{2N-2n}). \tag{A4}$$

The second sum can be performed exactly, and equals $(N+1)(1+\tau^{2N})-2(1-\tau^{2N+2})/(1-\tau^2)$. The first sum can be rearranged by means of the expansion $(1+z)/(1-z)^2 = \sum_{a=0}^{\infty} (2a+1)z^a$. We obtain

$$\chi_{(\infty),l} = p^2(1-\tau^2)^2 r^{l+1} \sum_{a=0}^{\infty} \frac{(2a+1)\tau^{4a}\tau^{(2a+1)l}}{(1-r\tau^{2a})^2(1-r\tau^{2a+2})^2}. \tag{A5}$$

This expression yields

$$\chi_{(\infty)}(q) = p^2 r(1-\tau^2)^2 \sum_{a=0}^{\infty} \frac{(2a+1)\tau^{4a}}{(1-r\tau^{2a})^2(1-r\tau^{2a+2})^2} \frac{1-(r\tau^{2a+1})^2}{1-2r\tau^{2a+1}\cos q+(r\tau^{2a+1})^2}. \tag{A6}$$

The analytic structure of $\chi_{(\infty)}(q)$ is very clearly read from this expression. The correlation function has an infinity of simple poles, located at $e^{\pm iq} = r\tau^{2a+1}$, labelled by the so-far abstract expansion index $a \geq 0$. Each pole originates in a component of $\chi_{(\infty),l}$ that decays exponentially as $(r\tau^{2a+1})^{|l|}$. Hence $\chi_{(\infty),l}$ can be considered as the superposition of an infinity of (lattice) Lorentzians. The term with $a=0$ is decaying the slowest; hence it determines the correlation length

$$\zeta = e^{-1/\xi} = r\tau. \tag{A7}$$

As temperature goes to zero, the sequence of poles merges at $e^{\pm iq} = r$, and produces a more complex behaviour. The $T=0$ correlation function can be studied either directly from (A6), or from the following expression, that can be derived from (A4):

$$\chi_{(\infty),l} = \frac{1}{3} p^2 r^{l-1} \sum_{N=0}^{\infty} \frac{N(N^2-1)r^N}{(N+l)^2} \quad l \geq 0. \tag{A8}$$

The value of $\chi_{(\infty),0}$ at $T=0$ is easily deduced from this last equation

$$\chi_{(\infty),0} = \frac{1}{3} \left(1 + \frac{p^2 \ln p}{r} \right). \tag{A9}$$

The rest of the calculation is more lengthy. After a good deal of simplification, our final expression for the function $\chi_{(\infty)}(q)$ at $T=0$ becomes

$$\chi_{(\infty)}(q) = \frac{p}{2 \sin^2 q/2} + \frac{p^2}{8r \sin^4 q/2} [\Phi(r e^{iq}) + \Phi(r e^{-iq}) - 2\Phi(r)] \tag{A10}$$

with the definition

$$\Phi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = - \int_0^z \ln(1-t) \frac{dt}{t}. \tag{A11}$$

The zero-temperature limit (A10) cannot be expressed in terms of elementary functions. The function $\Phi(z)$ has a singularity at $z=1$, of the form

$$\Phi_{\text{sg}}(z) \sim (1-z)[\ln(1-z)-1]. \tag{A12}$$

Hence the correlation function $\chi_{(\infty)}(q)$ has logarithmic branch cuts at $e^{\pm iq} = r$, which are the accumulation points of the above-mentioned poles. Nevertheless its value at zero wavevector is simple: $\chi_{(\infty)}(0) = (2-p-p^2)/(6p)$. More interesting is the decay of the correlation at large distance. This behaviour is dominated by the singularity (A12), which yields

$$\chi_{(\infty),l} \sim \frac{2r}{p^2} \frac{r^{|l|}}{l^2}. \tag{A13}$$

This expression can also be derived directly from (A8). The expected exponential decay is corrected by the squared inverse distance. This effect, which occurs only at $T=0$ strictly, has been overlooked in [4].

The zero-temperature correlation function has a scaling form in the $p \rightarrow 0$ limit. For small p , the typical distance between two sites with non-zero random fields diverges as $1/p$, and hence the wavevector should be scaled as p . This property is in fact valid for any finite value of H , as explained in § 5.5. In the present case, if we insert $q = pQ$ in (A10), the expression simplifies for $p \rightarrow 0$. We obtain

$$\lim_{p \rightarrow 0} p \chi_{(\infty)}(pQ) = X(Q) = \frac{2}{Q^4} [Q^2 + \ln(1 + Q^2) - 2Q \tan^{-1} Q]. \quad (\text{A14})$$

This last expression can also be derived by replacing the sum in the RHS of (A8) by an integral. It is worth noticing that, even in this scaling limit, the correlation function is highly non-trivial. It is particularly clear from the behaviour of the scaling function $X(Q)$ for small and large Q , given in (5.28a) and (5.29a), that this is not a simple function, such as the combination of a Lorentzian and a Lorentzian squared.

The above-mentioned logarithmic branch points are still present in $X(Q)$, at $Q = \pm i$. They are responsible for the following long-distance decay of the correlation in the $p \rightarrow 0$ limit: $\chi_{(\infty),l} \sim 2 e^{-p|l|} / (pl)^2$, which is clearly the scaling form of (A13).

References

- [1] Fishman S and Aharony A 1979 *J. Phys. C: Solid State Phys.* **12** L729
- [2] Yoshizawa H, Cowley R A, Shirane G, Birgeneau R J, Guggenheim H J and Ikeda H 1982 *Phys. Rev. Lett.* **48** 438
- [3] Derrida B, Vannimenus J and Pomeau Y 1978 *J. Phys. C: Solid State Phys.* **11** 4749
- [4] Grinstein G and Mukamel D 1983 *Phys. Rev. B* **27** 4503
- [5] Derrida B and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 2641
- [6] Bruinsma R and Aeppli G 1983 *Phys. Rev. Lett.* **50** 1494
- [7] Györgyi G and Rujan P 1984 *J. Phys. C: Solid State Phys.* **17** 4207
- [8] Normand J M, Mehta M L and Orland H 1985 *J. Phys. A: Math. Gen.* **18** 621
- [9] Fonk Y and Hilhorst H J 1987 *J. Stat. Phys.* **49** 1235
- [10] Nieuwenhuizen Th M and Luck J M 1986 *J. Phys. A: Math. Gen.* **19** 1207
- [11] Nieuwenhuizen Th M 1983 *Physica* **120A** 468
- [12] Nieuwenhuizen Th M 1984 *Physica* **125A** 197
- [13] Nieuwenhuizen Th M 1984 *Phys. Lett.* **103A** 333
- [14] Nieuwenhuizen Th M 1987 *Europhys. Lett.* **4** 1109
- [15] Kogon H S and Wallace D J 1981 *J. Phys. A: Math. Gen.* **14** L527
- [16] Nattermann T and Villain J 1988 *Phase Transitions* **11** 5